Putnam training problems

2017 - Set 5

Problem 1 *Find* $\lim_{n\to\infty} \left|\sin(\pi\sqrt{n^2+n+1})\right|$

Problem 2 *Determine if there is an infinite sequence* $a_1, a_2, \ldots$ *such that for all positive integers* m *we have*

$$a_1^m + a_2^m + a_3^m + \ldots = m$$

Problem 3 *Prove that* $\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1} dx = \frac{1}{2}$.

Problem 4 *Suppose that the real numbers* $a_0, a_1, \ldots, a_n$ *and* x*, with* $0 < x < 1$*, satisfy*

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y *with* $0 < y < 1$ *such that*

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Problem 5 *Prove that the sequence* $\{a_n\}_{n\in\mathbb{N}}$ *given by*

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{\ldots + \sqrt{n}}}}}}$$

converges.

Problem 6 *Let* a_1, b_1, c_1 *be real numbers. We construct sequences* $\{a_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ *using the recursive formula*

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \frac{b_n + c_n}{2} \quad c_{n+1} = \frac{c_n + a_n}{2}.$$

Prove that each of these sequences converges and find their limits.

Problem 7 *Let* $|\varepsilon| < 1$ *be a real number. Prove that for any fixed* t *in* $\mathbb{R}$ *the equation* $t = x - \varepsilon \sin x$ *has a unique solution* x*.*

Problem 8 *For* p *a positive integer find the value of*

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$$

Problem 9 *Prove that*

$$\sum_{n=1}^{9999} \frac{1}{(\sqrt{n}+\sqrt{n+1})(\sqrt[4]{n}+\sqrt[4]{n+1})} = 9.$$

Problem 10 *Does the limit*

$$\lim_{x\to\pi/2} \sin(x)^{\frac{1}{\cos(x)}}$$

exist?