Putnam training problems

2017 - Set 3 hints

Problem 1 *Find all natural numbers n such that $n!$ ends in exactly 1000 zeroes.*

Hint for problem 1 Show that the larges k such that 5^k divides $n!$ is $\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5^2} \rfloor + \ldots$.

Problem 2 *Prove that if n is a natural number then 2^n does not divide $n!$.*

Hint for problem 2 Show that if n has k digits 1 in its binary expansion, then the larges power of 2 that divides $n!$ is 2^{n-k}.

Problem 3 *Prove that if $n > 1$ is a natural number then $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is not an integer.*

Hint for problem 3 Let k be the only power of two such that $2^k \leq n < 2^{k+1}$. Use this to show that when you simplify the sum to a single reduced fraction $\frac{a}{b}$, then a is odd while b is a multiple of 2^k.

Second hint for problem 3 Let p be the greatest prime such that $p \leq n$. Show that there is only one multiple of p among $1, 2, \ldots, n$ using Bertrand's postulate (there is at least one prime number between k and $2k$ for all k). Why does this help?

Problem 4 *Find all solutions in the positive integers to the equation $x^{x+y} = y^{y-x}$.*

Hint for problem 4 Show that y is a mulitple of x, so $y = nx$. Show that the equation reduces to $x^2 = n^{n-1}$. Then, n is either odd or a perfect square. This should give you two families of solutions.

Problem 5 *Determine if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that is strictly increasing, and for which $f(1) = 2$ and $f(f(n)) = f(n) + n$ for all $n \in \mathbb{N}$.*

Hint for problem 5 Prove and use Zeckendorf's theoerem to construct such a function. The theorem says that for an $n \in \mathbb{N}$ there is a unique way of writing n as a sum of different, non-consecutive Fibonacci numbers and not using F_1.

Problem 6 *Find all positive integers n such that $\frac{n^3-3n+4}{2n-1}$ is an integer.*

Hint for problem 6 Notice that the fraction T in the problem is an integer if an only if $8T$ is an integer. You can then reduce the degree of the numerator. Repeat this trick as needed.

Problem 7 *Prove that there is a Fibonacci number whose last ten digits are zero.*

Hint for problem 7 Consider the pairs (F_n, F_{n+1}) modulo 10^{10}. If two such pairs (F_a, F_{a+1}) and (F_b, F_{b+1}) are congruent, what about (F_{a-1}, F_a) and (F_{b-1}, F_b)?

Problem 8 *Prove that for every positive integer k there is a Fibonacci number with at least k distinct prime divisors.*

Hint for problem 8 Prove that every pair of consecutive Fibonacci numbers are relatively prime. Prove that if $a \neq 2$, then F_a divides F_b if and only if a divides b.

Problem 9 *Show that the equation*

$$x^{2008} + 2008! = 21^y$$

does not have solutions in the integers.

Hint for problem 9 Prove that x is a multiple of 21. Then give a bound on y and see that 2008! does not have enough factors to satisfy the equation.

Problem 10 *For any prime p prove that there are infinitely many multiples of p whose last ten digits are different.*

Hint for problem 10 Use the chinese remainder theorem modulo 10^10 if $p \neq 2, 5$.

Problem 11 *Prove that the equation*

$$2^x + 3 = z^3$$

has no solutions in the integers.

Hint for problem 11 Use modulo 7 and then modulo 13.

Problem 12 *Let n be a positive integer. Find the greatest common denominator of $n! + 1$ and $(n + 1)!$.*

Hint for problem 12 Show that this common denominator has to divide $n+1$. Use Wilson's theorem to finish.

Problem 13 *Prove that if x is an integer, then x^2+1 has no divisors of the form $4k+3$.*

Problem 14 *Assume that the result is false. If p is a prime of the form $4k+3$, compute $(-1)^{(p-1)/2}$ modulo p in two ways.*

Problem 15 *Prove that there is an infinite number of prime of the form $4k+1$.*

Hint for problem 15 If there is only a finite list $p_1, \ldots, p_n$ of such primes, consider $m = (p_1 p_2 \cdots p_n)^2 + 1$.