Putnam training problems

2017 - Set 7

Problem 1 *Show that there are no non-zero integer solutions to the equation*

$$a^2 + b^2 = 3(x^2 + y^2)$$

Hint for problem 1 Look at the equation modulo 3. If there is a non-zero equation, consider one that minimizes $a^2 + b^2 + x^2 + y^2$.

Problem 2 *Let S be a finite set of points in the plane such that for any two points in S, there is at least one more point in S that is collinear with them. Prove that there is a line containing all of S.*

Hint for problem 2 Suppose there is a solution where not all the points are different. Consider the set of all lines spanned by at least two of the points in S. Let d be the smallest from a line in that set to a point in S that is not contained in the line. Construct a smaller distance.

Problem 3 *Suppose that in a party with $n \geq 4$ people, we know that for each group of four persons there are either three of them who know each other or three of them such that no two of them know each other. Prove that we can split the people into two rooms, one where every two persons know each other, and another where no two persons know each other.*

Hint for problem 3 Proceed inductively. Suppose that you can split all persons $\{1, \ldots, k-1\}$ into two rooms: A where they all know each other and B where they all don't know each other. Let (A, B) be the splitting such that the sum of the number of persons in A that don't know person k and the number of persons in B that know person k is minimal. If k cannot be added to either room, show that you can modify (A, B) to reduce the sum mentioned.

Problem 4 *Prove that every graph G has a bipartite subgraph with at least half as many edges as G.*

Hint for problem 4 Choose a splitting of the graph into two sets U, V with the largest number of edges with one endpoint in U and one endpoint in V. If a vertex v in V has less than half its neighbors in U, what can you do?

Problem 5 *Prove that every convex set in the plane of area one is contained in a rectangle of area two.*

Hint for problem 5 Let p, q be the two points furthest apart in the set. Show that the set is contained between the lines perpendicular to the segment p, q that go through p, and q. Then, close it to form a rectangle.

Problem 6 *Prove that a graph with n vertices and no triangles has at most $\frac{n^2}{4}$ edges.*

Hint for problem 6 Look at v the vertex with highest degree a. Show that the graph has at most $a \cdot (n-a)$ edges.

Problem 7 *Let S be a closed, bounded set in the plane such that for any two points a, b in S, S contains one of the two arcs from a to b of the circle with diameter ab. Prove that S is a disk.*

Hint for problem 7 Proceed with a similar argument to Problem 5.

Problem 8 *Let S be a closed, bounded set in space such that any plane cuts of S are circles. Prove that S is a sphere.*

Hint for problem 8 Proceed with a similar argument to Problem 5.

Problem 9 *There are n identical cars in a circular track. Together they have just enough gas for a complete lap. Show that there is at least one car that complete a lap by collecting the gas of the other cars as it goes around.*

Hint for problem 9 Use induction on n.

Problem 10 *There are n blue points and n red points in the plane, no three of the $2n$ points are collinear. Show that we can pair the blue and red points using n segments that do no cross each other.*

Hint for problem 10 For each pairing of the blue and red points into colorful segments, consider the one with minimal sum of length of the segments.

Problem 11 *In space, several solid balls with unit radius are given (a finite number). On the surface of each ball the points where none of the other balls are visible are marked. Show that the total area marked is equal to the surface area of one of the balls.*

Hint for problem 11 For each direction v, consider the furthest point in direction v in each ball. How many can be marked?

Problem 12 *A group of 23 friends wants to play soccer. They will choose a referee and split into two teams of 11 persons each. They notice that, regardless of who is the referee, they can always split into two teams whose total weight is the same. If all of them have integer weights, prove that they all have the same weight.*

Hint for problem 12 If there is a solution where not all weights are equal, assume that this solution has a minimal sum of weights. Look at the solution modulo 2 and seek a way to reduce it.