Putnam training problems

2017 - Set 8 hints

Problem 1 *A Hadamard Matrix is a an $n \times n$ matrix such that every entry is 1 or -1 and every pair of columns is orthogonal. Prove that if A is a Hadamard matrix and $n > 2$ then n is a multiple of 4.*

Hint for problem 1 Since the condition means $H^-1 = (1/n)H^T$, every two rows are also orthogonal. Notice that by multiplying rows by -1 if needed, you may assume that the first column has only entries 1. How many entries of the second column can be 1 and how many -1? What happens for the third column?

Problem 2 *Do there exist $n \times n$ matrices A, B such that $AB - BA = I_n$?*

Hint for problem 2 Look at the trace of $AB - BA$.

Problem 3 *Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Find the determinant of the $n \times n$ matrix*

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^{n-1} \end{pmatrix}$$

Hint for problem 3 Proceed by induction on n (first see what happens with $n = 2$). Replace every α_n by a variable x. The determinant becomes a polynomial on x. What is the degree? Find $n - 1$ roots of the polynomials. Compute $p(0)$.

Problem 4 *Find the determinant of the matrix*

$$\begin{pmatrix} (1+x^2)^2 & (1+xy)^2 & (1+xz)^2 \\ (1+xy)^2 & (1+y^2)^2 & (1+yz)^2 \\ (1+xz)^2 & (1+yz)^2 & (1+z^2)^2 \end{pmatrix}$$

Hint for problem 4 Factorize the matrix using (slight modifications of) the matrices

$$\begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$$

Problem 5 *Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$ where*

$$a_{ij} = \begin{cases} (-1)^{|i-j|} & \text{if } i \neq j \\ 2 & \text{if } i = j. \end{cases}$$

Hint for problem 5 Use row operations to reduce the matrix to something simpler.

Problem 6 *Let $A = (a_{ij})$ be an $n \times n$ matrix such that*

$$\sum_{j=1}^{n} |a_{ij}| < 1$$

for all i. Prove that $I_n - A$ is invertible.

Hint for problem 6 Remember that for real numbers, if $|x| < 1$, then $\frac{1}{1-x} = 1 + x + x^2 + \ldots$. Prove that the series of matrices $I_n + A + A^2 + A^3 + \ldots$ converges.

Problem 7 *Prove that in* $\mathbb{R}^d$ *there is no set of* $d+2$ *vectors whose pairwise angles are obtuse.*

Hint for problem 7 Prove that for any $d+2$ points in $\mathbb{R}^d$, there is a partition of them into two sets such that their convex hulls intersect. If p is such a point, prove that $\langle p, p \rangle < 0$, where $\langle \cdot, \cdot \rangle$ denotes the dot product.

Problem 8 *Let* X *be a set of* n *elements and* $X_1, \ldots, X_{n+1}$ *be non-empty subsets of* X*. Prove that we can find find two non-empty families* $\mathcal{A}, \mathcal{B}$ *of the* X_i *who don't share any* X_i *but such that* $\cup\mathcal{A} = \cup\mathcal{B}$*.*

Hint for problem 8 Assign to each set a vector in $\mathbb{R}^n$ where the entries are 1 or 0 depending on which elements it has. Use a linear dependence between the vectors to produce $\mathcal{A}$ and $\mathcal{B}$.

Problem 9 *There are n persons in a town. They decide to form clubs (persons may be in more than one club). They notice that every club has an odd number of persons and any two clubs share an even number of persons. What is the largest number of clubs there can be?*

Hint for problem 9 Assign to each club a vector in $\mathbb{R}^n$ wit entries 1 or 0 depending on which persons are members of the club. Look at the dot products between these vectors modulo 2 to show that they are linearly independent. What does that tell you about the number of clubs?