4 Class 4 - Construction of the natural numbers

The book mentions that if we believe in the Peano axioms (i.e. we believe that we cannot prove a contradiction using them), then we can construct a set that satisfies the axioms of the real numbers. This is what we call *relative consistency.*

Given a set of axioms, if we cannot prove a contradiction we say that they are *consistent.* The result the book refers to is the following.

Theorem 4.1. *If the Peano axioms are consistent, then the axioms of the real numbers are consistent.*

Proving this theorem exceeds the objective of the course[5]. However, if we are studying the real numbers, we should be able to construct the set of natural numbers. In other words, we will prove the following theorem.

Theorem 4.2. *If the axioms of the real numbers are consistent, then the Peano axioms are consistent.*

Definition 4.3. *We say a subset A of the real numbers is inductive if it satisfies the following properties*

- $1 \in A$,
- *for every a in A, $a+1 \in A$*

Examples.

- The set of all real numbers $\mathbb{R}$ is inductive
- The set of all positive real numbers $A = \{x \in \mathbb{R} : x > 0\}$ is inductive (prove it!)
- The set of all real numbers greater than 1 is not inductive.

Definition 4.4. *We define the set $\mathbb{N}$ of natural numbers as the intersection of all inductive sets*

$$\mathbb{N} = \bigcap_{A \text{ is inductive}} A$$

Notice that $\mathbb{N}$ is well defined as there are inductive sets. Moreover, $\mathbb{N}$ is not empty as every inductive set contains 1, so $1 \in \mathbb{N}$. Also notice that $\mathbb{N}$ is inductive. If an element a is in every inductive set, so is $a+1$.

The reason for this seemingly artificial construction[6] is to be able to prove that induction works.

Recall the Peano axioms

- **(N1)** There is an element 1 in $\mathbb{N}$

[5]From a formal point of view, it's also not very helpful. The proof of this theorem consists of assuming that the Peano axioms work and then constructing as set that satisfies the axioms of the real numbers. However, this construction may also satisfy additional properties. This is very bad if our goal is to study the consequences of the axioms of the real numbers!

[6]If you don't think this construction is artificial, you should take a course in set theory.

- **(N2)** Every element n in $\mathbb{N}$ has a successor $s(n)$ in $\mathbb{N}$.
- **(N3)** 1 is not the successor of any element in $\mathbb{N}$.
- **(N4)** For any elements n and m in $\mathbb{N}$, $n = m$ if and only if they have the same successor.
- **(N5)** A subset of $\mathbb{N}$ which contains 1, and which contains $s(n)$ whenever it contains a natural number n, must equal $\mathbb{N}$.

Theorem 4.5. *The set $\mathbb{N}$ we just constructed satisfies the Peano axioms.*

Proof.

- **(N1)** We mentioned before that 1 (the number 1 from the real numbers) is in $\mathbb{N}$. This is what we will use as 1 (the number 1 from the Peano axioms).
- **(N2)** If we define $s(n) = n + 1$, then every element in $\mathbb{N}$ has its successor since $\mathbb{N}$ is inductive.
- **(N3)** Since the set $\mathbb{R}^+$ of all positive real numbers is inductive, then $\mathbb{N} \subset \mathbb{R}^+$. However, $0 \notin \mathbb{R}^+$, so 0 is not an element of $\mathbb{N}$. This meanse that 1 is not the successor of any element in
- **(N4)** $n = m$ if and only if $n + 1 = m + 1$.
- **(N5)** Let A be a subset of $\mathbb{N}$ with the mentioned properties. By definition A is inductive, so $\mathbb{N} \subset A$. So we have $\mathbb{N} \subset A$ and $A \subset \mathbb{N}$. This means that $A = \mathbb{N}$, as desired.

$\square$

This is the set of natural integers. Using them, we get the usual notation for the real numbers. In other words, we define $2 = 1 + 1$, $3 = 2 + 1$, and so on. Since we defined the natural numbers using our strange construction, we don't know yet that they have the usual arithmetic properties. However, we can prove them by induction.

Theorem 4.6. *The sum of two natural numbers is a natural number.*

Proof. We will prove by induction on n that every element of the form $m + n$ for any natural m is also a natural number.

Basis of induction If $n = 1$, then $m + 1$ is a natural number, as desired.

Hypothesis of induction We assume that $m + n$ is a natural number.

Inductive set We want to prove that $m + (n + 1)$ is a natural number. However, $m + (n + 1) = (m + n) + 1$. Since $m + n$ is a natural number, this is also a natural number. $\square$

Theorem 4.7. *The product of two natural numbers is a natural number*

Proof. Left as homework. $\square$

Constructing the integers and rational numbers is now easy

Definition 4.8. *The set of integers is defined as $\mathbb{Z} = \{a \in \mathbb{R} : a = 0, a \in \mathbb{N} \text{ or } -a \in \mathbb{N}\}$.*

This is our commonly known set $\cdots - 3, -2, -1, 0, 1, 2, 3 \ldots$

Definition 4.9. *The set of rational numbers is defined as*

$$\mathbb{Q} = \left\{ q \in \mathbb{R} : q = \frac{a}{b}, \text{ where } a, b \in \mathbb{Z}, b \neq 0 \right\}$$

One motivation to construct the integers and rational sets is to get an *algebraic closure* of the natural numbers. For instance, if we only use the natural numbers we don't have additive inverses, so it is convenient to work with the integers if we seek that. The sum and product of integers is also an integer (with our construction, you can prove this!), but we don't have multiplicative inverses. That's why the rational numbers come into play.

One of the interesting features of the rational numbers is that it is *dense* in $\mathbb{Q}$ (we will see later that it is also dense in $\mathbb{R}$).

Theorem 4.10. *For any two rational numbers $a, b \in \mathbb{Q}$, if $a \leq b$ there exists $c \in \mathbb{Q}$ such that $a \leq c \leq b$.*

Proof. Consider $c = \frac{a+b}{2}$. Since $a \leq b$ this means that $2a \leq a + b \leq 2b$. If we divide by two we get the desired inequality. $\square$

The main disadvantage of the rational numbers is that they are insufficient to develop a rich theory of continuous functions. In particular, even though the rational numbers are dense, there are still gaps (this will become clear later on). The last axiom of the real numbers is to avoid this.