Putnam training problems

2017 - Set 1 hints

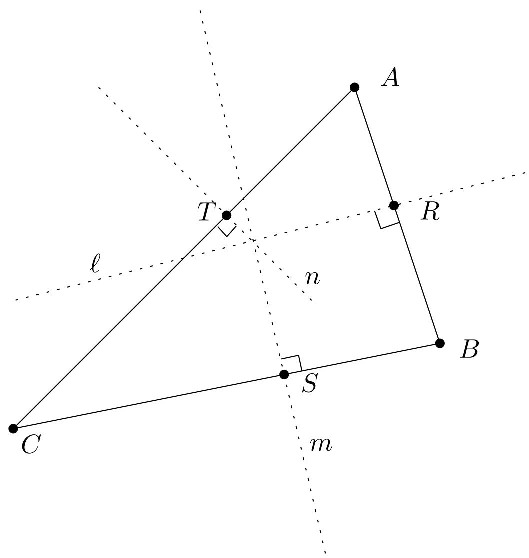
Problem 1. Hint 1: Given a set $A \subset \{1, 2, \ldots, 2n\}$ of $n+1$ numbers, you can write every element $m \in A$ as $m = 2^a b$ where a is a non-negative integer and b is odd. How many possible values can b take?

Hint 2: Suppose that A is as above and no element in A divides some other element in A. Let $a = \min A$. Show that $2a \leq 2n$. Consider the set A' obtained by replacing a by $2a$. Is it possible for there to be two elements in A' such that one divides the other? How many times can you repeat this process?

Problem 2 hint. Consider the numbers of the form $3^a 7^b$ modulo 20. What residues can they have?

Problem 3 hint. First prove that every term in the sequence is non-negative. Notice that $x_n = e^{x_n} - e^{x_{n+1}}$. Use this to write $x_1 + x_2 + \ldots + x_m$ as a telescopic sum. Prove that the desired sum is convergent. What does that tell you about $\lim_{m\to\infty} x_m$?

Problem 4 hint. If the three lines are concurrent, use Pythagoras' theorem. If the equation holds but we don't know if the lines are concurrent, let P be the intersection of n and m. Let R' be the orthogonal projection of P onto the line AB. Why must we have $R = R'$?



Problem 5 hint. Let $\triangle ABC$ be a triangle of maximal area with vertices in S. For every point $D \in S$ we know that the area of $\triangle ABD$ is at most that of $\triangle ABC$. What does that tell you about the position of D? Repeat with all sides of $\triangle ABC$

Problem 6 hint. Use the fact that if x, y are to independent random variables, then $E(xy) = E(x)E(y)$, where $E(\cdot)$ denotes the expectation. Also, for any k random variables $x_1, \ldots, x_k$ (independent or not), we have $E(x_1 + \ldots + x_k) = E(x_1) + \ldots E(x_k)$. For this problem, expand the determinant of $A - A^t$ as a sum of products $\prod_{i=1}^{2n}(a_{i\sigma(i)} - a_{\sigma(i)i}$, where $\sigma : [2n] \to [2n]$ is a permutation. The first fact will tell you that many of these products are zero. For the rest, try to compute their expectation explicitly.

Problem 7 hint. Let T be the number of pairs of persons which are living in the same apartment. After a fight, show that T decreases. Can this continue forever?