Putnam training problems

2017 - Set 4 hints

Problem 1 *Show that a among any $n+2$ integers either there are two whose sum is divisible by $2n$ or there are two whose difference is divisible by $2n$.*

Hint for problem 1 Consider the $n+1$ sets $i, 2n-i$ for $i = 0, 1, \dots, n$ and use the pigeonhole principle.

Problem 2 *Show that for any set of n integers, there is always a non-empty subset whose sum is divisible by n.*

Hint for problem 2 If $a_1, \dots, a_n$ are your numbers, consider

$$\begin{aligned}
b_1 &= a_1 \\
b_2 &= a_1 + a_2 \\
b_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
b_n &= a_1 + a_2 + a_3 + \ldots + a_n
\end{aligned}$$

Problem 3 *Show that among any nine points with integer coordinates in space there is always two of them such that the segment that joins them contains at least one more integer point.*

Hint for problem 3 Looks at your points $\bar{a} = (x, y, z)$ modulo 2. How many different possibilities do you have?

Problem 4 *All sides and diagonals of an octagon are colored either red or blue. Show that there are at least seven monochromatic triangles with vertices among the vertices of the octagon.*

Hint for problem 4 You can actually show that there are eight monochromatic triangles (seven can be shown with a very precise case analysis). Define a "colored angle" as a a triple x_1, x_2, x_3 of vertices of your octagon such that x_1x_2 and x_2x_3 are of different colors. Every non-monochromatic triangle uses exactly two colored angles. Find an upper bound for the number of colored angles.

Problem 5 *Each 1×1 cell of a 100×100 grid are colored with one of four possible colors in such a way that every row and every column has exactly 25 cells of each color. Show that there are four cells of the grid forming an axis-parallel rectangle which are colored with four different colors.*

Hint for problem 5 Count the number of pairs (a, b) where a, b are both 1×1 cells of the same row but painted of different columns.

Problem 6 *Show that if the plane is colored with 3 colors we can always find two points at distance 1 which have the same color.*

Hint for problem 6 Show that every equilateral triangle of side length 1 must be painted with all three colors. Show that every two points at distance $\sqrt{3}$ must be painted with the same color.

Problem 7 *Let $\triangle$ be an equilateral triangle in the plane. Show that if the points in the plane are colored red and blue, then there is either a segment of length one whose endpoints are both blue, or there is a translated copy of $\triangle$ whose vertices are all red.*

Hint for problem 7 Number the vertices of $\triangle$ as $1, 2, 3$. For each x in the plane consider $x + \triangle$ to be the translate of $\triangle$ such that vertex 1 is in x. If $x + \triangle$ does not have all vertices red, label x with A, B or C according to how $x + \triangle$ is labeled. By problem 6 there are two points at distance 1 with the same label.

Problem 8 *Let $n = ab + 1$, where a, b are positive integers. Show that in any ordered list of n different numbers there is either a sublist of length $a + 1$ which is increasing or a sublist of length $b + 1$ which is decreasing.*

Hint for problem 8 for each x in the sequence, assign to it a pair of positive integers (a_x, b_x) such that a_x is the length of the longest sublist which is increasing as ends in x, while b_x is the length of the longest sublist which is decreasing and ends in x. Use the pigeonhole principle to get a contradiction.

Problem 9 *Let A be a set of $n^2 + 1$ lines in the plane, colored red, such that no two of them are parallel. Show that we can always find a vertical black line ℓ such that we can find a set of $n + 1$ red lines whose intersections are all on the left side of ℓ, and a set of $n + 1$ red lines whose intersections are all on the right side of ℓ.*

Hint for problem 9 Use the previous problem

Problem 10 *Let $A_1, \ldots, A_{2n}$ be pairwise different subsets of a set with n elements. Find the minimum value of*

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|},$$

where we consider $A_{2n+1} = A_1$.

Hint for problem 10 Prove that $\frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}}| \leq \frac{1}{2}$ for all i.

Problem 11 *Let n and k be positive integers. Joe and José are going to write lists of integers. Joe writes down all the lists $a_1, \ldots, a_k$ such that $|a_1| + |a_2| + \ldots + |a_k| \leq n$. José writes down all the lists $b_1, \ldots, b_n$ such that $|b_1| + |b_2| + \ldots + |b_n| \leq k$. Prove that Joe and José wrote down the same number of lists.*

Hint for problem 11 Prove that the number of lists with exactly r non-zero terms is $\binom{n}{r}\binom{k}{r}2^r$ for both Joe and José.