Putnam training problems

2017 - Set 5 hints

Problem 1 *Find* $\lim_{n\to\infty} \left|\sin(\pi\sqrt{n^2+n+1})\right|$

Hint for problem 1 Prove that $\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-\left(n+\frac{1}{2}\right)\right)=0.$

Problem 2 *Determine if there is an infinite sequence* $a_1, a_2, \ldots$ *such that for all positive integers* m *we have*

$$a_1^m + a_2^m + a_3^m + \ldots = m$$

Hint for problem 2 If we have $a_i \leq i$ for all i, who that the left hand side is a decreasing function of m. What happens if some $a_i > i$?

Problem 3 *Prove that* $\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}dx = \frac{1}{2}.$

Hint for problem 3 Prove that $\lim_{x\to 0} x^x = 1.$

Problem 4 *Suppose that the real numbers* $a_0, a_1, \ldots, a_n$ *and* x*, with* $0 < x < 1$*, satisfy*

$$\frac{a_0}{1-x}+\frac{a_1}{1-x^2}+\cdots+\frac{a_n}{1-x^{n+1}}=0.$$

Prove that there exists a real number y *with* $0 < y < 1$ *such that*

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Hint for problem 4 Use the fact that $\frac{1}{1-u} = 1 + u + u^2 + \ldots$ as long as $|u| < 1$. Use the mean value theorem if there are no zeroes for $a_0 + a_1 y + \cdots + a_n y^n$.

Problem 5 *Prove that the sequence* $\{a_n\}_{n\in\mathbb{N}}$ *given by*

$$a_n = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{\ldots+\sqrt{n}}}}}}$$

converges.

Hint for problem 5 Notice that it is sufficient to show that the sequence is bounded. To do this, consider a new sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $b_1 = k$ for some constant k and $b_{n+1} = b_n^2 - n$. Prove that the equation $k \geq a_n$ is equivalent to $b_{n+1} \geq 0$. Find a suitable k that allows you to prove this by induction.

Problem 6 *Let* a_1, b_1, c_1 *be real numbers. We construct sequences* $\{a_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ *using the recursive formula*

$$a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \frac{b_n + c_n}{2} \quad c_{n+1} = \frac{c_n + a_n}{2}.$$

Prove that each of these sequences converges and find their limits.

Hint for problem 6 Let $x_n = \max\{|a_n - b_n|, |b_n - c_n|, |c_n - a_n|\}$. Prove that $\lim_{n\to\infty} x_n = 0$. Then, notice that $a_n - \frac{a_n+b_n+c_n}{3} = \frac{a_n-b_n}{3} + \frac{a_n-c_n}{3}$.

Problem 7 *Let* $|\varepsilon| < 1$ *be a real number. Prove that for any fixed* t *in* $\mathbb{R}$ *the equation* $t = x - \varepsilon \sin x$ *has a unique solution* x*.*

Hint for problem 7 Use the mean value theorem and the intermediate value theorem.

Problem 8 *For* p *a positive integer find the value of*

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$$

Hint for problem 8 Compare $1^p + 2^p + \ldots + n^p$ with $\int_0^n x^p dx$ and with $\int_1^{n+1} x^p dx$.

Problem 9 *Prove that*

$$\sum_{n=1}^{9999} \frac{1}{(\sqrt{n}+\sqrt{n+1})(\sqrt[4]{n}+\sqrt[4]{n+1})} = 9.$$

Hint for problem 9 Multiply the numerator and denominator by $\sqrt[4]{n+1}-\sqrt[4]{n}$.

Problem 10 *Does the limit*

$$\lim_{x\to\pi/2} \sin(x)^{\frac{1}{\cos(x)}}$$

exist?

Hint for problem 10 Use L'Hopital.