Putnam training problems

2017 - Set 7

Problem 1 *Show that there are no non-zero integer solutions to the equation*

$$a^2 + b^2 = 3(x^2 + y^2)$$

Problem 2 *Let S be a finite set of points in the plane such that for any two points in S, there is at least one more point in S that is collinear with them. Prove that there is a line containing all of S.*

Problem 3 *Suppose that in a party with $n \geq 4$ people, we know that for each group of four persons there are either three of them who know each other or three of them such that no two of them know each other. Prove that we can split the people into two rooms, one where every two persons know each other, and another where no two persons know each other.*

Problem 4 *Prove that every graph G has a bipartite subgraph with at least half as many edges as G.*

Problem 5 *Prove that every convex set in the plane of area one is contained in a rectangle of area two.*

Problem 6 *Prove that a graph with n vertices and no triangles has at most $\frac{n^2}{4}$ edges.*

Problem 7 *Let S be a closed, bounded set in the plane such that for any two points a, b in S, S contains one of the two arcs from a to b of the circle with diameter ab. Prove that S is a disk.*

Problem 8 *Let S be a closed, bounded set in space such that any plane cuts of S are circles. Prove that S is a sphere.*

Problem 9 *There are n identical cars in a circular track. Together they have just enough gas for a complete lap. Show that there is at least one car that complete a lap by collecting the gas of the other cars as it goes around.*

Problem 10 *There are n blue points and n red points in the plane, no three of the $2n$ points are collinear. Show that we can pair the blue and red points using n segments that do no cross each other.*

Problem 11 *In space, several solid balls with unit radius are given (a finite number). On the surface of each ball the points where none of the other balls are visible are marked. Show that the total area marked is equal to the surface area of one of the balls.*

Problem 12 *A group of 23 friends wants to play soccer. They will choose a referee and split into two teams of 11 persons each. They notice that, regardless of who is the referee, they can always split into two teams whose total weight is the same. If all of them have integer weights, prove that they all have the same weight.*