Putnam training problems

2017 - Set 6

Problem 1 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function such that* $f\left(\frac{x_1+x_2}{2}\right) \geq \frac{f(x_1)+f(x_2)}{2}$ *for all* $x_1, x_2 \in \mathbb{R}$*. Prove that for all positive integers* n *and all* $x_1, \ldots, x_n \in \mathbb{R}$ *we have*

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \geq \frac{f(x_1) + \ldots + f(x_n)}{n}.$$

Hint for problem 1 Use induction. Prove that if the statement holds for n, then it holds for $2n$ and for $n-1$.

Problem 2 *Let* $a_1, a_2, \ldots, a_n$ *be real numbers such that* $a_i \geq 1$ *for all* i*. Prove that*

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \ldots + \frac{1}{1+a_n} \geq \frac{n}{1 + \sqrt[n]{a_1 a_2 \ldots a_n}}.$$

Hint for problem 2 Use induction. Prove the statement for $n = 2$ and prove that if the statement holds for n, then it holds for $2n$ and for $n-1$.

Problem 3 *Find*

$$\min_{a,b \in \mathbb{R}} \max\{a^2 + b, b^2 + a\}.$$

Hint for problem 3 Prove that $(a^2 + b) + (b^2 + a) \geq -\frac{1}{2}$.

Problem 4 *Prove that if* a, b, c *are non-negative numbers such that* $(1+a)(1+b)(1+c) = 8$*, then*

$$abc \leq 1.$$

Hint for problem 4 Prove that for any non-negative a, b, c, we have $(1+a)(1+b)(1+c) \geq (1 + \sqrt[3]{abc})^3$.

Problem 5 *Prove that if* a, b, c *are real positive numbers, then*

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a} \geq \frac{3}{2}.$$

Hint for problem 5 Use the rearrangemenet inequality.

Problem 6 *Prove that if* $a_1, \ldots, a_n$ *are real numbers, then*

$$\sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + \ldots + a_n}{n}$$

Hint for problem 6 Use the rearrangement inequality n times.

Second hint for problem 6 Use the Cauchy-Scharts inequality when one of the lists consists only of 1.

Problem 7 *Prove that if* a, b, c *are positive real numbers and* n *is a positive integers, then*

$$\frac{a^n}{b+c} + \frac{b^n}{a+b} + \frac{c^n}{a+b} \geq \frac{a^{n-1} + b^{n-1} + c^{n-1}}{2}.$$

Hint for problem 7 Use the rearrangement inequality.

Problem 8 *Prove that for* $a_1, \ldots, a_n, b_1, \ldots, b_n$ *non-negative real numbers, we have*

$$\sqrt[n]{a_1 \ldots a_n} + \sqrt[n]{b_1 \ldots b_n} \leq \sqrt[n]{(a_1 + b_1) \ldots (a_n + b_n)}$$

Hint for problem 8 Prove, using the same type of induction as in the first problems, that for non-negative $c_1, \ldots, c_n$, we have

$$(1 + c_1) \ldots (1 + c_n) \geq (1 + \sqrt[n]{c_1 \ldots c_n})^n.$$

Why can we assume that $a_i = 1$ for all i.

Hint for problem 8 Show that you can assume that $a_i + b_i = 1$ for all i. Then, use the (Geometric Mean)-(Arithmetic Mean) inequality on $\sqrt[n]{a_1 \dots a_n}$ and on $\sqrt[n]{b_1 \dots b_n}$.

Problem 9 *If a, b, c are the sides of a triangle, prove that*

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3$$

Hint for problem 9 Since a, b, c are the sides of a triangle, there are non-negative real numbers r, s, t such that $a = r + s$, $b = s + t$, $c = t + r$. Use this substittiution.

Problem 10 *Suppose a, b, c are the sides of a triangle. Prove that*

$$a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \leq 3abc.$$

Hint for problem 10 Since a, b, c are the sides of a triangle, there are non-negative real numbers r, s, t such that $a = r + s$, $b = s + t$, $c = t + r$. Use this substittiution.

Problem 11 *Prove that if $a_1, \dots, a_n$ are positive real numbers, then*

$$\frac{a_1 + \dots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}.$$

Hint for problem 11 Write the problem as

$$(a_1 + \dots + a_n)\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right) \geq n^2.$$

Use Cauchy-Schwartz or the rearrangement inequality.

Problem 12 *Let $x_1, \dots, x_n$ be positive real numbers and $s = x_1 + \dots + x_n$. Prove that*

$$\frac{s}{s-x_1} + \frac{s}{s-x_2} + \dots + \frac{s}{s-x_n} \geq \frac{n^2}{n-1}$$

Hint for problem 12 Use the previous problem.