

Putnam training problems
2017 - Set 6

Problem 1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f\left(\frac{x_1+x_2}{2}\right) \geq \frac{f(x_1)+f(x_2)}{2}$ for all $x_1, x_2 \in \mathbb{R}$. Prove that for all positive integers n and all $x_1, \dots, x_n \in \mathbb{R}$ we have

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{f(x_1) + \dots + f(x_n)}{n}.$$

Problem 2 Let a_1, a_2, \dots, a_n be real numbers such that $a_i \geq 1$ for all i . Prove that

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \dots + \frac{1}{1+a_n} \geq \frac{n}{1 + \sqrt[n]{a_1 a_2 \dots a_n}}.$$

Problem 3 Find

$$\min_{a,b \in \mathbb{R}} \max\{a^2 + b, b^2 + a\}.$$

Problem 4 Prove that if a, b, c are non-negative numbers such that $(1+a)(1+b)(1+c) = 8$, then

$$abc \leq 1.$$

Problem 5 Prove that if a, b, c are real positive numbers, then

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{b+a} \geq \frac{3}{2}.$$

Problem 6 Prove that if a_1, \dots, a_n are real numbers, then

$$\sqrt{\frac{a_1^2 + \dots + a_n^2}{n}} \geq \frac{a_1 + \dots + a_n}{n}$$

Problem 7 Prove that if a, b, c are positive real numbers and n is a positive integers, then

$$\frac{a^n}{b+c} + \frac{b^n}{a+b} + \frac{c^n}{a+b} \geq \frac{a^{n-1} + b^{n-1} + c^{n-1}}{2}.$$

Problem 8 Prove that for $a_1, \dots, a_n, b_1, \dots, b_n$ non-negative real numbers, we have

$$\sqrt[n]{a_1 \dots a_n} + \sqrt[n]{b_1 \dots b_n} \leq \sqrt[n]{(a_1 + b_1) \dots (a_n + b_n)}$$

Problem 9 If a, b, c are the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3$$

Problem 10 Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \leq 3abc.$$

Problem 11 Prove that if a_1, \dots, a_n are positive real numbers, then

$$\frac{a_1 + \dots + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}.$$

Problem 12 Let x_1, \dots, x_n be positive real numbers and $s = x_1 + \dots + x_n$. Prove that

$$\frac{s}{s-x_1} + \frac{s}{s-x_2} + \dots + \frac{s}{s-x_n} \geq \frac{n^2}{n-1}$$